

# Open Gromov-Witten Invariants from the Augmentation Polynomial

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A conjecture of Aganagic and Vafa [4] relates the open Gromov-Witten theory of  $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1)$  to the augmentation polynomial of Legendrian contact homology. We describe how to use this conjecture to compute genus zero, one boundary component open Gromov-Witten invariants for Lagrangian submanifolds  $L_K \subset X$  obtained from the conormal bundles of knots  $K \subset S^3$ . This computation is then performed for two non-toric examples (the figure-eight and three-twist knots). For  $(r, s)$  torus knots, the open Gromov-Witten invariants can also be computed using Atiyah-Bott localization. Using this result for the unknot and the  $(3, 2)$  torus knot, we show that the augmentation polynomial can be derived from these open Gromov-Witten invariants.

## 1. Introduction

Gromov-Witten theory has famously benefited from its connections with string dualities, first with mirror symmetry [6, 25], and more recently, large  $N$  duality [10, 22]. Beginning with [17], for toric manifolds, Gromov-Witten invariants associated to maps of closed surfaces have also been systematically computed using localization [7, 11, 16]. Although the analogous constructions in open Gromov-Witten theory are not rigorously defined in general, many of the same computational tools (such as mirror symmetry and Atiyah-Bott localization) can still be applied. In contrast to the closed theory, open Gromov-Witten theory also possesses many direct relationships with knot theory. Large  $N$  duality relates Chern-Simons theory on  $S^3$  to Gromov-Witten theory on  $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1)$  via the conifold transition. Wilson loops in Chern-Simons theory on  $S^3$  are also related to the HOMFLY polynomials of knots  $K \subset S^3$  [24]. This relationship equates the colored HOMFLY polynomials of knots  $K$  with generating functions for open Gromov-Witten invariants of  $X$  with Lagrangian boundary  $L_K$  obtained from the conormal bundle  $N_K^* \subset T^*S^3$ , and has been checked for torus knots in [8].

Recently, it has also been suggested that open Gromov-Witten theory is related to another type of knot invariant arising in Legendrian contact homology [1, 4]. For a knot  $K \subset S^3$ , Legendrian contact homology associates a dga  $\mathcal{A}(\Lambda_K)$  to its unit conormal bundle  $\Lambda_K \subset U^*S^3$ . The unit conormal bundle  $\Lambda_K \approx T^2$  is a Legendrian submanifold of the unit cotangent bundle  $U^*S^3$  (a contact manifold), and the differential on  $\mathcal{A}(\Lambda_K)$  is obtained, roughly speaking, from counts of maps of holomorphic disks to  $\Lambda_K \times \mathbb{R}$ . An augmentation of  $\mathcal{A}(\Lambda_K)$  is a dga map  $\epsilon : \mathcal{A}(\Lambda_K) \rightarrow \mathbb{C}$ , where  $\mathbb{C}$  is interpreted as a dga with trivial differential. The moduli space of such augmentations is described by an equation  $A_K(x, p, Q) = 0$ , where  $x$ ,  $p$ , and  $Q$  are generators for  $H_2(U^*S^3, \Lambda_K)$ .  $A_K$  is called the augmentation polynomial of the knot  $K$ . (More detailed accounts of Legendrian contact homology can be found in [21] and [1]).

In [4], Aganagic and Vafa conjecture that the moduli space of open Gromov-Witten invariants on  $(X, L_K)$  is encoded by the augmentation polynomial  $A_K$ . The goal of this paper is to use this conjecture to compute Gromov-Witten invariants and augmentation polynomials. Crucially, the augmentation polynomial can be computed for non-toric knots, and hence, this method can be applied in scenarios where Atiyah-Bott localization cannot be used. On the other hand, for  $(r, s)$  torus knots, open Gromov-Witten invariants are known from localization, and this data provides another means of obtaining the augmentation polynomial of  $K$ .

### 1.1. Organization of the paper

This paper is organized in the following way. Section 2 reviews mirror symmetry for open Gromov-Witten invariants, and describes Aganagic and Vafa's conjecture relating mirror symmetry and the augmentation polynomial. Section 3 applies this conjecture to compute open Gromov-Witten invariants in two non-toric examples. Finally, Section 4 performs the reverse of this computation: the open Gromov-Witten invariants associated to torus knots are used to recover the corresponding augmentation polynomials. (Note that for  $(r, s)$  torus knots, open Gromov-Witten invariants in framing  $rs$  have been computed directly via localization in [8]).

## 2. Open String Mirror Symmetry and the Augmentation Polynomial

Recently, new developments in knot theory and open topological string theory have uncovered connections between open Gromov-Witten theory and

knot theory [1, 4, 8, 22]. The subject of interest in this note is open Gromov-Witten theory for Lagrangian submanifolds of  $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1)$ . Let  $\Sigma$  be a genus-zero Riemann surface with one boundary component. Denote by  $K_{d,w}$  the open Gromov-Witten invariant associated to a stable map  $f : \Sigma \rightarrow X$  with Lagrangian boundary conditions on a Lagrangian submanifold  $L \subset X$ , where  $d = [f(\Sigma)] \in H_2(X, L)$  and  $w = [f(\partial\Sigma)] \in H_1(L)$ . This section describes a technique for computing  $K_{d,w}$  using a mirror symmetry conjecture [1, 4].

$X$  can be obtained by symplectic reduction on  $\mathbb{C}^4$ . Let  $(z_1, z_2, z_3, z_4)$  be coordinates for  $\mathbb{C}^4$ , and let  $S^1$  act on  $\mathbb{C}^4$  with weights  $(1, 1, -1, -1)$ ; then

$$(1) \quad X \cong \left\{ |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = r \right\} / S^1,$$

where  $r \in \mathbb{R}_{>0}$ . The coordinates on the base  $\mathbb{P}^1$  are  $z_1$  and  $z_2$ , and the  $z_3, z_4$  coordinates parametrize the fiber. According to the Strominger-Yau-Zaslow conjecture for noncompact  $X$  [3, 23],  $X$  is a special Lagrangian fibration over a base  $B \cong \mathbb{R}^3$ , with generic fibers  $L \cong T^2 \times \mathbb{R}$ . In these coordinates, the base  $B$  and the special Lagrangian fibers  $L$  are easy to describe. The base  $B$  is the image of  $X$  under the moment map  $z_i \mapsto |z_i|^2$ , and the fibers  $L$  are given by the equations

$$(2) \quad \begin{aligned} |z_2|^2 - |z_4|^2 &= c_1, \\ |z_3|^2 - |z_4|^2 &= c_2, \\ \text{Arg}(z_1 z_2 z_3 z_4) &= 0, \end{aligned}$$

where  $c_1, c_2 \in \mathbb{R}$ . For generic values of  $c_1, c_2$ ,  $L$  has topology  $T^2 \times \mathbb{R}$ ; however, as either  $c_1 \rightarrow 0$  or  $c_2 \rightarrow 0$ , the topology of the fibers degenerates to two copies of  $S^1 \times \mathbb{R}^2$ . This critical locus along which the fibers degenerate forms a trivalent graph in  $B$ , corresponding to the “edges” of the moment polytope. The moment polytope and special Lagrangian fibers are depicted in Figure 1.

### 2.1. The Mirror of $\mathcal{O}_{\mathbb{P}^1}(-1, -1)$

The construction of [3, 13] gives the mirror manifold  $\hat{X}$  to  $X$  in terms of a dual Landau-Ginsburg theory. For  $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1)$ , the mirror equation is

$$\hat{X} = \left\{ uv = y_2 + y_3 + y_4 + \frac{y_3 y_4}{y_2} e^{-t} \right\} / \mathbb{C}^*,$$

where  $u, v \in \mathbb{C}$ ,  $y_j \in \mathbb{C}^*$ ,  $t = r + i\theta$  is the complexified Kähler parameter, and the  $\mathbb{C}^*$  acts diagonally on the  $y_j$ 's. The SYZ conjecture asserts that the

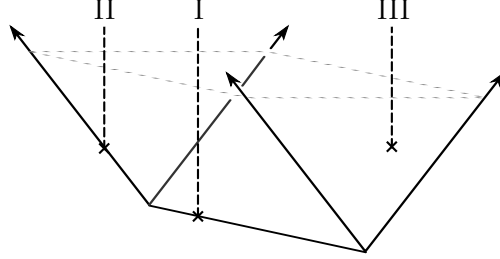


Figure 1: The moment polytope and special Lagrangian fibers of  $X$ . The moment polytope of  $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1)$  is its image  $\pi(X) \subset \mathbb{R}^4$  under the moment map  $\pi(z_i) = |z_i|^2$ . The images of the special Lagrangian fibers of  $X$  in the moment polytope are vertical lines, which can intersect  $X$  three ways: along the base  $\mathbb{P}^1$  (type I), along an exterior leg of the polytope (type II), or on a face (type III). Lagrangian fibers of type III have topology  $T^2 \times \mathbb{R}$ , corresponding to  $c_1 \neq 0$  and  $c_2 \neq 0$  in (2). Fibers of type I and II have topology  $S^1 \times \mathbb{R}^2$ , corresponding to either  $c_2 = 0$  or  $c_1 = 0$ , respectively.

mirror  $\hat{X}$  is the moduli of the special Lagrangian fibers of  $X$ . The choice of coordinate patch for the  $y_j$ 's determines which “phase” of Lagrangian fibers is being described. This coordinate choice is explained in further detail in [2]. In this note, the relevant coordinates are the  $y_2 = 1$  patch. In addition, to achieve later agreement with conventions from knot contact homology, the following change of coordinates will be used:

$$Q := e^t, \quad x := -y_3/Q, \quad p := -y_4/Q.$$

Then, in this patch and coordinate system, the mirror manifold is

$$(3) \quad \hat{X} = \{uv = 1 - Qx - Qp + Qxp\}.$$

**Example (Open Gromov-Witten invariants via mirror symmetry).**

Consider Lagrangian fibers  $L$  of type I. These Lagrangians have topology  $S^1 \times \mathbb{R}^2$ , and intersect the base  $\mathbb{P}^1$  along the  $S^1$ . In the mirror, the moduli space of such Lagrangian fibers is the Riemann surface  $S \subset \hat{X}$  defined by setting  $uv = 0$ :

$$S = \{1 - Qx - Qp + Qxp = 0\}.$$

Mirror symmetry equates the periods of certain differential forms on  $\hat{X}$  to generating functions for open Gromov-Witten invariants. In this case, the prediction of mirror symmetry is that (up to constant factors in  $x$ )

$$(4) \quad \int \lambda = \sum_{d,w} K_{d,w} Q^d x^w,$$

where  $\lambda := -\log p \frac{dx}{x}$  is a one-form along  $S$  defined by solving  $1 - Qx - Qp + Qxp = 0$  for  $p$ , and  $K_{d,w}$  is the genus-0, degree- $d$ , winding  $w$  open Gromov-Witten invariant with boundary on  $L$ . In terms of  $x$  and  $Q$ ,

$$\begin{aligned} -\log p(x; Q) &= \log(Q) + \log\left(\frac{1-x}{1-Qx}\right) \\ &= \log(Q) + \sum_{n=1}^{\infty} \frac{1}{n} (-1 + Q^n) x^n, \end{aligned}$$

so (4) asserts that

$$\sum_{d,w} K_{d,w} Q^d x^w = \sum_{n=1}^{\infty} \frac{1}{n^2} (-1 + Q^n) x^n.$$

Hence

$$K_{d,w} = \begin{cases} -\frac{1}{w^2}, & d = 0; \\ \frac{1}{d^2}, & d = w; \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this case,  $K_{d,w}$  has also been computed directly using localization in [15], and these results agree in framing 0.

Open Gromov-Witten invariants are also conjectured to satisfy certain integrality requirements [3, 18]. For genus-zero, one-boundary-component invariants, the requirement is that

$$(5) \quad K_{d,w} = \sum_{n|d \text{ and } n|w} \frac{1}{n^2} N_{d/n, w/n},$$

where  $N_{d,w} \in \mathbb{Z}$ . In terms of the generating function for  $K_{d,w}$ , this is

$$\sum_{d,w} K_{d,w} Q^d x^w = \sum_{n>0} \sum_{d,w} \frac{1}{n^2} N_{d,w} Q^{dn} x^{wn}.$$

In the previous example,  $N_{0,1} = -1$ ,  $N_{1,1} = 1$ , and  $N_{d,w} = 0$  for all other  $d, w$ .

## 2.2. Knots and the Conifold Transition

The Lagrangian submanifolds described by (2) have a very specific geometry. A natural question is, what are the open Gromov-Witten invariants associated to Lagrangians with a different geometry? One way of obtaining such Lagrangians is through the conifold transition [8, 22]. The manifold  $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1)$  can also be obtained as the resolution of the conifold singularity in  $\mathbb{C}^4$ —it is given by the equations

$$\begin{aligned}xz - yw &= 0, \\x\lambda &= w\rho, \\y\lambda &= z\rho\end{aligned}$$

where  $((x, y, z, w), [\lambda : \rho]) \in \mathbb{C}^4 \times \mathbb{P}^1$ . The conifold singularity  $xz - yw = 0$  is the limit of the hypersurface

$$Y_\mu := \{xz - yw = \mu\} \subset \mathbb{C}^4$$

as  $\mu \rightarrow 0$ . For  $\mu \neq 0$ ,  $Y_\mu$  is symplectomorphic to the cotangent bundle  $T^*S^3$ . The zero section  $S_\mu \cong S^3 \subset Y_\mu$  is the fixed locus of the antiholomorphic involution  $(x, y, z, w) \mapsto (\bar{z}, -\bar{w}, \bar{x}, -\bar{y})$ , and is described by the equation  $|x|^2 + |y|^2 = \mu$ . Away from the zero sections, the conifold transition gives a symplectomorphism between  $T^*S^3$  and  $X$  [8]. Thus, a Lagrangian submanifold of  $T^*S^3$  which does not intersect the zero section  $S^3$  is symplectomorphic to a Lagrangian submanifold of  $X$  which also does not intersect the zero section.

Knots  $K \subset S^3$  are a source of Lagrangian submanifolds in  $T^*S^3$ : the conormal bundle  $N_K^* \subset T^*S^3$  is Lagrangian with topology  $S^1 \times \mathbb{R}^2$ . In order to obtain a Lagrangian submanifold of  $X$  from  $N_K^*$ ,  $N_K^*$  must first be moved off of the zero section. This can be done by choosing a lift of  $K$ , as described in [8]. The image of the shifted conormal bundle will be a Lagrangian submanifold  $L_K \subset X$ , as depicted in Figure 2.

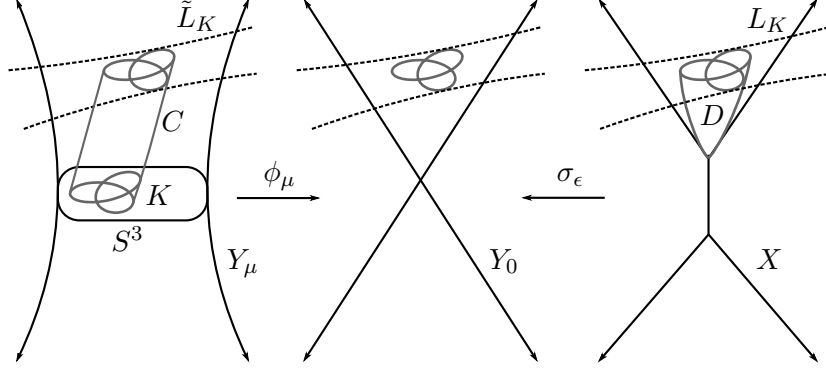


Figure 2: The conifold transition and knots.

The Lagrangian  $\tilde{L}_K \subset Y_\mu \cong T^*S^3$  is constructed by shifting the conormal bundle of a knot  $K \subset S^3$  off of the zero section. This lift introduces a holomorphic cylinder  $C$  connecting the knot on  $S^3$  to its image in  $\tilde{L}_K$ .  $Y_0$  is the conifold singularity  $xz - yw = 0$  in  $\mathbb{C}^4$ . The map  $\phi_\mu : Y_\mu \rightarrow Y_0$  is a symplectomorphism away from the zero section, so  $\phi_\mu(\tilde{L}_K)$  is a Lagrangian submanifold of  $Y_0$ .  $X \cong \mathcal{O}_{\mathbb{P}^1}(-1, -1)$  is the small resolution of the conifold singularity, and  $\sigma_\epsilon : X \rightarrow Y_0$  is the corresponding natural map. In fact, there are a family of such symplectomorphisms, where  $\epsilon$  parametrizes the symplectic form on the zero section  $\mathbb{P}^1 \subset X$ . Hence,  $L_K := \sigma_\epsilon^{-1} \circ \phi_\mu(\tilde{L}_K)$  is a Lagrangian submanifold of  $X$ . The holomorphic disk  $D$  is the image of  $C$  under the conifold transition.

### 2.3. Open Gromov-Witten Invariants and the Augmentation Polynomial

For torus knots, the corresponding open Gromov-Witten invariants of  $(X, L_K)$  have been computed directly using localization in [8], and using mirror symmetry mirror symmetry in [5, 14]. However, the mirror symmetry approach of [5] does not readily generalize to non-toric knots. In this subsection, a recent conjecture of Aganagic and Vafa [4] is applied to compute open Gromov-Witten invariants via an analogous mirror symmetry computation. This approach uses the augmentation polynomial from Legendrian contact homology. In contrast to localization and the method of [5], this method can be

applied for any knot whose augmentation polynomial is known—including non-toric knots.

The Aganagic-Vafa conjecture can be motivated by observing that the mirror to  $\hat{X}$  given in (3) can be written as

$$\hat{X} = \{uv = A(x, p; Q)\},$$

where  $A(x, p; Q) = 1 - Qx - Qp + Qxp$  is the augmentation polynomial of the unknot in Legendrian contact homology [9, 21]. The moduli of Lagrangian submanifolds with topology  $S^1 \times \mathbb{R}^2$  was described by the zero locus

$$\{A(x, p; Q) = 0\} \subset \hat{X}.$$

Moreover, the image of a shifted conormal bundle to the unknot under the conifold transition described in Section 2.2 is exactly one of these Lagrangian fibers [12, 22]. Hence, one might speculate that the moduli of Lagrangian submanifolds  $L_K \subset X$  obtained from the conormal bundles of other knots  $K \subset S^3$  are given by the Riemann surface

$$\{A_K(x, p; Q) = 0\},$$

where  $A_K(x, p; Q)$  is the augmentation polynomial of the knot  $K$ .

Aganagic and Vafa then conjecture that for each choice of knot, there is a corresponding mirror  $\hat{X}_K$  describing the moduli space of Lagrangian fibers with geometry  $L_K$  along the singular locus, and that  $\hat{X}_K$  is described by

$$\hat{X}_K := \{uv = A_K(x, p; Q)\}.$$

In addition to the apparent coincidence between the unknot's augmentation polynomial and the mirror  $\hat{X}$ , there are also physical arguments for this conjecture coming from the connections between topological string theory, Chern-Simons theory, and the HOMFLY polynomial [1, 4, 14, 20]. The augmentation polynomial  $A_K$  can be identified with the classical limit of the  $Q$ -deformed quantum A-polynomial, which satisfies an elimination condition on the Chern-Simons partition function in the presence of a Wilson loop coming from  $K$ . This Chern-Simons partition function can in turn be obtained from the colored HOMFLY polynomials of  $K$ , and is identified with the open Gromov-Witten generating function under the conifold transition.

For the purposes of computing genus-zero open Gromov-Witten invariants, the application of this conjecture is straightforward: as in Section 2.1,



the open Gromov-Witten generating function is equated with the integral of a differential form:

$$\sum_{d,w} K_{d,w} Q^d x^w = \int -\log p(x; Q) \frac{dx}{x}.$$

However,  $p(x; Q)$  is now determined by the equation

$$A_K(x, p; Q) = 0.$$

The main difficulty is that  $A_K(x, p; Q)$  may be a high-order polynomial in  $p$ , so it is not always feasible to find analytic solutions for  $p$ . Even for torus knots, this can be an obstacle: for an  $(r, s)$  torus knot,

$$(6) \quad \deg p = \binom{r+s}{s}, \quad \deg x = \frac{\deg p}{r+s}$$

are the maximum degrees [14].

Fortunately, to compute  $K_{d,w}$  for a given  $d$  and  $w$ , only a series solution for  $p$  is needed. Suppose that

$$(7) \quad p(x; Q) = \exp \left( - \sum_{n=0}^{\infty} W_n(Q) x^n \right),$$

where  $W_n(Q)$  is a polynomial in  $Q$  for  $n > 0$ , and  $W_0(Q)$  determines the overall scaling of  $p$ . (Note that in all considered examples,  $W_0(Q) = \log(Q)$ , i.e.,  $p(x; Q) \approx 1/Q + \dots$ ). Then, substitute this expression into

$$A_K(x, p(x; Q); Q) = 0.$$

The resulting expression will be a series in  $x$ , which can be solved by recursively finding the coefficients  $W_n(Q)$ —in general, the coefficient of  $x^n$  will be a polynomial function of  $\{W_k(Q)\}_{k \leq n}$ . The coefficients  $W_n(Q)$  are related to the  $K_{d,w}$  by

$$W_n(Q) = \sum_d n K_{d,n} Q^d.$$

For completeness, the integer invariants  $N_{d,w}$  can be similarly determined from the  $K_{d,w}$ : (5) can be re-written as

$$N_{d,w} = K_{d,w} - \sum_{\substack{n|d \text{ and } n|w \\ n>1}} \frac{1}{n^2} N_{d/n, w/n},$$

so by starting with  $N_{0,1} = K_{0,1}$ , successive  $N_{d,w}$  can be solved for. The following section uses this method to compute  $K_{d,w}$  and  $N_{d,w}$  for two non-toric knots.

### 3. Non-Toric Examples

As compared to Atiyah-Bott localization, one advantage of computing open Gromov-Witten invariants from the augmentation polynomial is that this technique does not require that the Lagrangian  $L$  is fixed by a torus action. Such Lagrangians can be obtained from the conormal bundles of non-toric knots. This section performs the computation of Section 2.3 for two non-toric knots: the  $4_1$  (figure-eight) knot and the  $5_2$  (three-twist) knot. In contrast to torus knots, these knots cannot be expressed as the link of the singularity  $x^r - y^s = 0$  in  $S^3$  for any  $r, s \in \mathbb{Z}$ . For non-toric knots, it is not currently known how to directly compute the open Gromov-Witten invariants  $K_{d,w}$  using localization. However, the invariants computed in the following examples have been checked to satisfy the integrality condition (5) for all  $w \leq 8$  and arbitrary  $d$ .

#### 3.1. The $4_1$ (figure-eight) knot

The  $4_1$  knot is the unique knot with crossing number 4, and is a twist knot obtained from two half-twists. According to [4, 12], the augmentation polynomial of the  $4_1$  knot in framing 0 is

$$\begin{aligned} A_K(x, p, Q) = & p^2 - Qp^3 + (Q^3p^5 - 2Q^3p^4 + 2Qp - Q)x \\ & + (-Q^5p^5 + 2Q^4p^4 - 2Q^3p + Q^2)x^2 \\ & + (Q^5p^3 - Q^5p^2)x^3. \end{aligned}$$

Rescaling  $x$  by  $x \mapsto Qx$ , one can obtain a series solution for  $-\log p$  using the method of section 2.3. The first few terms are

$$\begin{aligned} -\log p = & \log(Q) + (Q^3 - 2Q^2 + 2Q - 1)x \\ & + \left( \frac{5Q^6}{2} - 8Q^5 + 9Q^4 - 9Q^2 + 8Q - \frac{5}{2} \right) x^2 + \cdots \end{aligned}$$

The coefficients of this series solution determine the open Gromov-Witten invariants  $K_{d,w}$  and corresponding integer invariants  $N_{d,w}$ . For  $w \leq 4$   $0 \leq d \leq 3$ , these invariants are listed in Table 1. The integrality condition  $N_{d,w} \in$

Table 1:  $K_{d,w}$  and  $N_{d,w}$  for the  $4_1$  knot.

$K_{d,w}$	$K_{d,1}$	$K_{d,2}$	$K_{d,3}$	$K_{d,4}$	$N_{d,w}$	$N_{d,1}$	$N_{d,2}$	$N_{d,3}$	$N_{d,4}$
$K_{0,w}$	-1	$-\frac{5}{4}$	$-\frac{28}{9}$	$-\frac{165}{16}$	$N_{0,w}$	-1	-1	-3	-10
$K_{1,3}$	2	4	14	60	$N_{1,3}$	2	4	14	60
$K_{2,w}$	-2	$-\frac{9}{2}$	-25	-147	$N_{2,w}$	-2	-5	-25	-148
$K_{3,w}$	1	0	$\frac{173}{9}$	186	$N_{3,w}$	1	0	19	186

$\mathbb{Z}$  has also been verified for all  $w \leq 8$  and arbitrary  $d$ . Note that for  $w \leq 8$ ,  $K_{d,w} = N_{d,w} = 0$  for any  $d > 24$ .

### 3.2. The $5_2$ (three-twist) knot

The  $5_2$  knot is a twist knot obtained from three half-twists, and is one of two knots with crossing number 5 (the other being the  $(5, 2)$  torus knot). The augmentation polynomial for the  $5_2$  knot is

$$\begin{aligned}
A_K(x, p, Q) = & Q^2 p^8 - Q p^7 + x^4 (-p + 1) \\
& + (-Q^3 p^6 + 2Q^2 p^5 - Q p^4 - 2Q p^3 + 3Q p^2 - 3p^2 + 4p - 2) x^3 \\
& + (Q^4 p^8 - 3Q^3 p^7 - 4Q^3 p^6 + 5Q^2 p^6 + 3Q^2 p^5 \\
& + 6Q^2 p^4 - 3Q p^5 - 4Q p^4 + 3Q p^3 - 4Q p^2 - 3p^3 + 5p^2 - 3p + 1) x^2 \\
& + (-2Q^3 p^8 + 4Q^2 p^7 + 3Q^2 p^6 - 3Q p^6 - 2Q p^5 - p^4 + 2p^3 - p^2) x.
\end{aligned}$$

Rescaling  $x$  by  $x \mapsto x/Q$ , the method of 2.3 gives a series solution for  $-\log p$ . The first few terms are

$$\begin{aligned}
-\log p = & \log(Q) + (-Q^4 + 2Q^3 - Q) x \\
& + \left( \frac{11Q^8}{2} - 20Q^7 + 23Q^6 - 8Q^5 + 2Q^4 - 4Q^3 + \frac{3Q^2}{2} \right) x^2 + \dots
\end{aligned}$$

The corresponding  $K_{d,w}$  and  $N_{d,w}$  are obtained from the coefficients of this series solution. These invariants are listed in Table 2. As for the  $4_1$  knot, the integrality condition  $N_{d,w} \in \mathbb{Z}$  has also been checked for all  $w \leq 8$ , and it was again found that  $K_{d,w} = N_{d,w} = 0$  for any  $w \leq 8$  and  $d > 24$ .

Table 2:  $K_{d,w}$  and  $N_{d,w}$  for the  $5_2$  knot.

$K_{d,w}$	$K_{d,1}$	$K_{d,2}$	$K_{d,3}$	$K_{d,4}$	$N_{d,w}$	$N_{d,1}$	$N_{d,2}$	$N_{d,3}$	$N_{d,4}$
$K_{0,w}$	-1	$\frac{3}{4}$	$-\frac{10}{9}$	$\frac{35}{16}$	$N_{0,w}$	-1	1	-1	2
$K_{1,3}$	0	-2	4	-12	$N_{1,3}$	0	-2	4	-12
$K_{2,w}$	2	1	0	$\frac{27}{2}$	$N_{2,w}$	2	1	0	14
$K_{3,w}$	-1	-4	-12	8	$N_{3,w}$	-1	-4	-12	8

#### 4. Recovering the Augmentation Polynomial

The augmentation polynomial conjecturally contains all of the open Gromov-Witten invariants  $K_{d,w}$  for a given Lagrangian brane  $L_K$ . As seen in the previous section, open Gromov-Witten invariants can be extracted from the augmentation polynomial. For torus knots, the genus zero open Gromov-Witten invariants have also been computed directly via localization in [8, 19]. Explicitly,

$$(8) \quad K_{d,w}^{(r,s)} = (-1)^{d+1} \left( \frac{\prod_{k=0}^{d-1} (wr - k)(ws - k)}{wd! \prod_{k=1}^d (wr + ws - k)} \right) \left( \frac{\prod_{k=1}^{ws-1} (r + s - \frac{k}{w})}{w \prod_{k=0}^{ws-1} (s - \frac{k}{w})} \right),$$

where  $r > s > 0$  are coprime, and  $ws \geq d \geq 0$ . For  $d > ws$ ,  $K_{d,w}^{(r,s)} = 0$ . This section describes a method for obtaining the augmentation polynomial  $A_K$  when  $K$  is a torus knot, and implements this method for two examples (the unknot, and the  $(3, 2)$  torus knot).

The idea is to use (8) and (7) to obtain a system of linear equations on the coefficients of the augmentation polynomial. Write

$$(9) \quad A_K(x, p; Q) = \sum_{j,k} c_{jk} x^j (p(x; Q))^k,$$

where  $c_{jk}$  is a rational polynomial in  $Q$ . Recall that for torus knots, the degrees of  $x$  and  $p$  are given by (6). Let

$$p(x; Q) = \frac{1}{Q} \exp \left( - \sum_{d,w} w K_{d,w} Q^d x^w \right)$$

and  $W_n := \sum_{d \geq 0} n K_{d,n} Q^d$ . Then the coefficient of  $x^n$  in a series expansion of  $(p(x; Q))^k$  is  $P_n(k)/Q^k$ , where

$$P_n(k) := \sum_{\substack{i_1 + 2i_2 + \dots + ni_n = n \\ i_j \geq 0}} \left[ \prod_{j=1}^n \frac{(-1)^{i_j}}{i_j!} (kW_j)^{i_j} \right].$$

( $P_0(k) = 1$ ). Substituting this expression into (9) gives a power series in  $x$ :

$$\begin{aligned} A_K(x, p; Q) &= \sum_{j,k} c_{jk} \left( \frac{x^j}{Q^k} \right) \left( \sum_{n \geq 0} P_n(k) x^n \right) \\ &= \sum_{n \geq 0} \left( \sum_{j=0}^n \sum_{k \geq 0} \frac{P_{n-j}(k)}{Q^k} c_{jk} \right) x^n. \end{aligned}$$

In order for  $p(x; Q)$  to be a solution of  $A_K(x, p; Q) = 0$ , the coefficient of  $x^n$  for all  $n$  in the above expression must vanish. This gives a collection of linear equations in the  $c_{jk}$ :

$$(10) \quad \sum_{j=0}^n \sum_{k \geq 0} \frac{P_{n-j}(k)}{Q^k} c_{jk} = 0,$$

which can be solved to determine the  $c_{jk}$  up to an overall rescaling of the augmentation polynomial. (Note that such rescalings do not affect the solutions  $p(x; Q)$ , and hence do not affect the Gromov-Witten invariants). The following two examples implement this procedure for the unknot in framing 0 and the (3,2) torus knot in framing 6. For both knots, this method recovers the expected augmentation polynomial. However, for the (3,2) knot, due to computational complexity, some simplifying assumptions about the values of the  $c_{jk}$  are made.

#### 4.1. The unknot

For the unknot,  $\deg x = 1$ ,  $\deg p = 1$ , so there are four coefficients to solve for:  $c_{00}$ ,  $c_{01}$ ,  $c_{10}$ , and  $c_{11}$ . Here, the  $W_n$  have a simple expression:

$$W_n = \frac{1}{n} (Q^n - 1).$$

The first four linear equations from (10) are:

$$\begin{aligned} c_{00} + \frac{1}{Q}c_{01} &= 0, & (n = 0) \\ \frac{1}{Q}(1 - Q)c_{01} + c_{10} + \frac{1}{Q}c_{11} &= 0, & (n = 1) \\ \frac{1}{Q}(1 - Q)c_{01} + \frac{1}{Q}(1 - Q)c_{11} &= 0. & (n = 2, 3) \end{aligned}$$

(The  $n = 2$  and  $n = 3$  equations both simplify to the same expression.) With  $c_{00}$  as the free variable, these equations become

$$c_{01} = -Qc_{00}, \quad c_{10} = -Qc_{00}, \quad c_{11} = Qc_{00}.$$

Normalizing to  $c_{00} = 1$  yields the expected augmentation polynomial of the unknot in framing 0:

$$A_K(x, p; Q) = 1 - Qx - Qp + Qxp.$$

#### 4.2. The (3, 2) torus knot

The augmentation polynomial of the (3, 2) torus knot is

$$\begin{aligned} A_K(x, p; Q) &= 1 - Qp + (Q^5p^3 - Q^5p^4 + 2Q^5p^5 - 2Q^6p^5 - Q^6p^6 + Q^7p^7)x \\ &\quad + (Q^{10}p^{10} - Q^{10}p^9)x^2. \end{aligned}$$

For the (3, 2) torus knot,  $\deg p = 10$  and  $\deg x = 2$ . So,  $c_{jk} = 0$  for all  $j > 2$  and  $k > 10$ . Note that in general, one would need to consider at least  $(\deg p + 1)(\deg x + 1)$  equations to determine  $c_{jk}$ . Due to the increasing complexity of the equations involved, in this example the following simplifying assumptions will be made:  $c_{0k} = 0$  for  $k > 1$ ,  $c_{10} = c_{11} = c_{12} = 0$ ,  $c_{18} = c_{19} = c_{1,10} = 0$ , and  $c_{2k} = 0$  for  $k < 9$ . Thus, the remaining nine “unknown” coefficients are  $c_{00}$ ,  $c_{01}$ ,  $c_{13}$ ,  $c_{14}$ ,  $c_{15}$ ,  $c_{16}$ ,  $c_{17}$ ,  $c_{29}$ , and  $c_{2,10}$ . The first

three equations from (10) are:

$$\begin{aligned}
c_{00} + \frac{1}{Q}c_{01} &= 0, \\
(2Q^{-1} - 3 + Q)c_{01} + Q^{-3}c_{13} + Q^{-4}c_{14} + Q^{-5}c_{15} + Q^{-6}c_{16} + Q^{-7}c_{17} &= 0, \\
(23Q^{-1} - 62 + 59Q - 23Q^2 + 3Q^3)c_{01} + (3Q^{-1} - 9Q^{-2} + 6Q^{-3})c_{13} \\
+ (4Q^{-2} - 12Q^{-3} + 8Q^{-4})c_{14} + (5Q^{-3} - 15Q^{-4} + 10Q^{-5})c_{15} \\
+ (6Q^{-4} - 18Q^{-5} + 12Q^{-6})c_{16} + (7Q^{-5} - 21Q^{-6} + 14Q^{-7})c_{17} \\
+ Q^{-9}c_{29} + Q^{-10}c_{2,10} &= 0.
\end{aligned}$$

(For brevity, the remaining equations are omitted). Solving for the  $c_{ij}$  (with  $c_{00}$  as the free variable), one finds

$$\begin{aligned}
c_{01} &= -Qc_{00}, & c_{13} &= Q^5c_{00}, & c_{14} &= -Q^5c_{00}, & c_{15} &= (2Q^5 - 2Q^6)c_{00}, \\
c_{16} &= -Q^6c_{00}, & c_{17} &= Q^7c_{00}, & c_{29} &= -Q^{10}c_{00}, & c_{2,10} &= Q^{10}c_{00}.
\end{aligned}$$

By normalizing to  $c_{00} = 1$ , the expected coefficients of the augmentation polynomial,

$$\begin{aligned}
c_{01} &= -Q, & c_{13} &= Q^5, & c_{14} &= -Q^5, & c_{15} &= 2Q^5 - 2Q^6, \\
c_{16} &= -Q^6, & c_{17} &= Q^7, & c_{29} &= -Q^{10}, & c_{2,10} &= Q^{10},
\end{aligned}$$

are obtained.

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